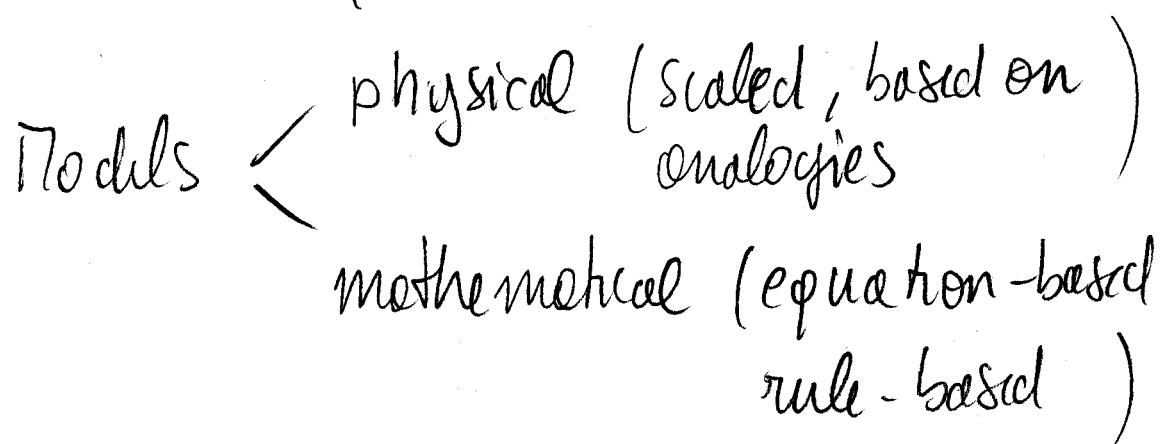


DYNAMICAL SYSTEMS

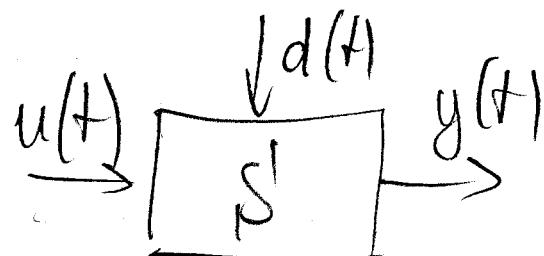
(1)

- DYNAMICAL SYSTEM MODELS

- Given a dynamical system S , a model of S is another dynamical system M_S that approximates some interesting aspects of its behaviour (possibly disregarding others)
- One can make experiments on M_S and infer how the actual S would behave



- In this course we'll focus on mathematical models

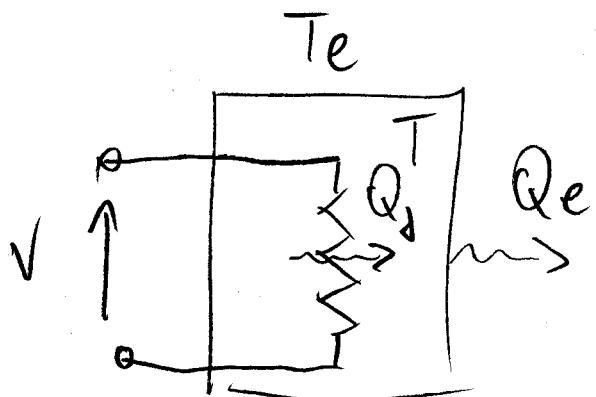


- Assuming $u(t), d(t), y(t)$ are functions of $t \in \mathbb{R}$ (i.e. $u(t) : \mathbb{R} \rightarrow \mathbb{R}^n$), S can be described by differential equations

DYNAMICAL SYSTEMS

(2)

- EXAMPLE ① Electrical heater*



Modelling assumptions

- Uniform temperature distribution
- constant heat transfer coefficient and specific heat coefficient
- (*) placed outdoors!

- Energy balance equation

$$\frac{dE}{dt} = Q_j - Q_e$$

$$E = c\pi T; \quad Q_j = \frac{V^2}{R}; \quad Q_e = KS(T - T_e)$$

$$c\pi \dot{T} = Q_j - KS(T - T_e)$$

- State variables: the variables that show up under derivative sign. Common symbol: x

$$x = T \quad \text{state variable}$$

$$u = Q_j \quad \text{input variable (includes } V \& R \text{)}$$

$$d = T_e \quad \text{disturbance (*)}$$

$$y_1 = T \quad y_2 = Q_e \quad \text{output variables}$$

DYNAMICAL SYSTEMS

(3)

- State-space description: the equations are solved to compute \dot{x}_i, y given u, d

$$\dot{T}(t) = -\frac{KS}{c\eta} T(t) + \frac{1}{c\eta} Q_j(t) + \frac{KS}{c\eta} T_e(t) \quad \begin{matrix} \text{state} \\ \text{equation} \end{matrix}$$

$$y_1 = T(t) \quad \begin{matrix} \text{output} \\ \text{equations} \end{matrix}$$

$$y_2 = Q_e(t) = KS(T(t) - T_e(t)) \quad \begin{matrix} \text{output} \\ \text{equations} \end{matrix}$$

- Solution of the system

Given : • the initial value of the state $T(t_0)$

• the time history of the inputs

$$Q_j(t), T_e(t) \quad t \geq t_0$$

a) Solve Cauchy's problem or find out $x(t)$

i.e. $T(t)$ (ODE integration)

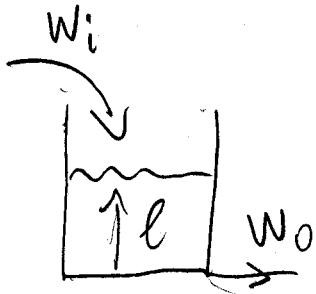
b) substitute in the output equations and compute the outputs

- Obviously, $T(\bar{t})$ depends on past values of $Q_j(t), T_e(t)$

DYNAMICAL SYSTEMS

4

- EXAMPLE ②: Tank with prescribed flows



- Modelling assumptions

- $\rho = \text{const}$
- cylindrical tank
cross-section A

- Mass balance equation

$$\frac{d\eta}{dt} = w_i - w_o$$

$$\eta = \rho A l$$

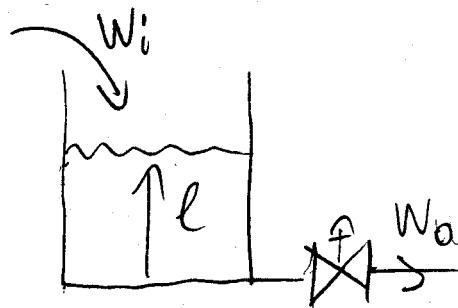
$$\rho A l \dot{l} = w_i - w_o$$

$$\left\{ \begin{array}{l} \dot{l} = \frac{1}{\rho A} (w_i - w_o) \\ y = l \end{array} \right. \quad \begin{array}{l} x = l \\ u = w_o \\ d = w_i \\ y = l \end{array}$$

DYNAMICAL SYSTEMS

(5)

- EXAMPLE (3) Tank with outlet valve



- Modelling assumptions
 - same as (2)

- Mass balance equation

$$\frac{d\eta}{dt} = w_i - w_o$$

$$\eta = \rho A l$$

$$w_o = K A_v \sqrt{l}$$

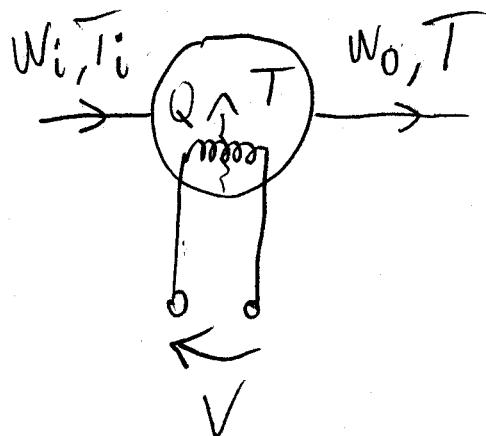
$$\rho A l = w_i - K A_v \sqrt{l}$$

$$\begin{cases} \dot{l} = \frac{1}{\rho A} (w_i - K A_v \sqrt{l}) \\ y = l \end{cases} \quad \begin{array}{l} x = l \\ d = w_i \\ u = A_v \\ y = l \end{array}$$

DYNAMICAL SYSTEMS

(6)

- EXAMPLE (4) Water Heater



- Modelling assumptions:

- $\rho = \text{const}$
- $C_p \approx C_V \approx c = \text{const}$
- Perfect thermal insulation
- Negligible wall heat capacity
- Uniform temp. distribution

- Mass balance equation

$$\frac{dM}{dt} = W_i - W_o$$

- Energy balance equation

$$\frac{dE}{dt} = W_i h_i - W_o h_o + Q$$

$$M = \rho V = \text{const}$$

$$E = M_e = M_c T$$

$$h_i = c T_i$$

$$h_o = c T$$

$$0 = W_i - W_o \Rightarrow W_i = W_o = W$$

$$M_c \dot{T} = W_c T_i - W_c T + Q$$

$$\dot{T} = \frac{W}{M} (T_i - T) + \frac{1}{M_c} Q$$

$$x = T$$

$$u = Q$$

$$d_1 = W$$

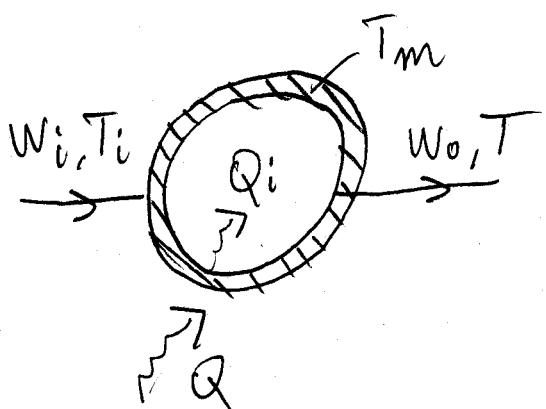
$$d_2 = T_i$$

$$y = T$$

DYNAMICAL SYSTEMS

(7)

- EXAMPLE (5) High-pressure boiler



- Modelling assumptions

- same as (4), except that the heat capacity of the walls is not negligible
- Q applied on external surface
- Uniform temp. distributions
- Constant heat transfer coefficient wall-fluid

- Mass balance equation

$$W_i = W_o = W \quad (\text{see } 4)$$

- Energy balance equation for the fluid

$$E = \text{Net}$$

$$\frac{dE}{dt} = W_{hi} - W_{ho} + Q_i$$

$$E_m = \bar{\rho}_m c_m \bar{T}_m$$

- Energy balance equation - walls

$$Q_i = KS(\bar{T}_m - T)$$

$$\frac{dE_m}{dt} = Q - Q_i$$

$$x_1 = T$$

$$Mc\dot{\bar{T}} = W_c\bar{T}_i - W_c\bar{T} + KS(\bar{T}_m - T)$$

$$x_2 = \bar{T}_m$$

$$\bar{\rho}_m c_m \dot{\bar{T}}_m = Q - KS(\bar{T}_m - T)$$

$$u = Q$$

$$\begin{cases} \dot{\bar{T}} = \frac{W}{M} (\bar{T}_i - \bar{T}) + \frac{KS}{Mc} (\bar{T}_m - \bar{T}) \\ \dot{\bar{T}}_m = \frac{1}{\bar{\rho}_m c_m} (Q - KS(\bar{T}_m - T)) \end{cases}$$

$$d_1 = W$$

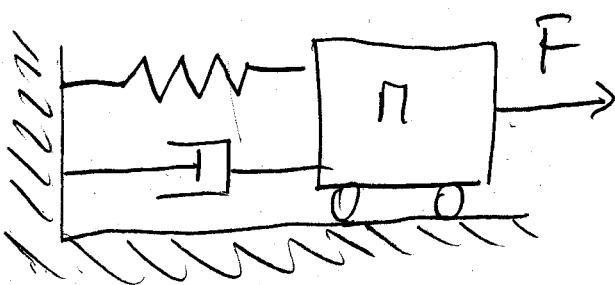
$$d_2 = T_i$$

$$y = \bar{T}$$

DYNAMICAL SYSTEMS

(8)

- EXAMPLE ⑥ Spring-mass oscillator



- Modelling assumptions:

- motion on the horizontal plane
- no friction

- Momentum balance equation

$$\frac{d}{dt} (Mv) = F - k_p - hv \quad v = p$$

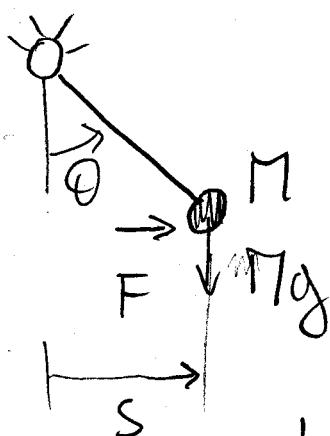
$$\left\{ \begin{array}{l} \dot{p} = v \\ \dot{v} = \frac{1}{M} (F - k_p - hv) \\ y = p \end{array} \right. \quad \begin{array}{l} x_1 = p \\ x_2 = v \\ u = F \end{array}$$

(harmonic oscillator)

DYNAMICAL SYSTEMS

(9)

- EXAMPLE (7) Simple pendulum



- Modelling assumption

- rigid massless link
- point mass

- Momentum balance equation

$$\frac{d}{dt} (Ml^2 \dot{\theta}) = Fl \cos\theta - Mg l \sin\theta - hw$$

$$\omega = \dot{\theta}$$

- State space eq's

$$x_1 = \theta \quad x_2 = \omega$$

$$u = F \quad y = S$$

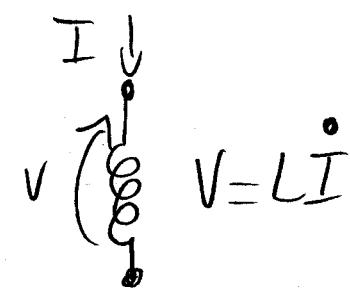
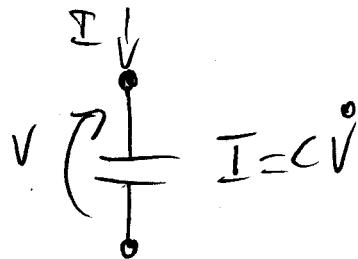
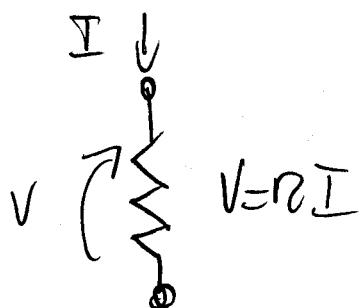
$$\begin{cases} \dot{\theta} = \omega \\ \ddot{\omega} = \frac{F}{Ml} \cos\theta - \frac{g}{l} \sin\theta - \frac{h}{Ml^2} \omega \end{cases}$$

$$y = l \sin\theta$$

DYNAMICAL SYSTEMS

(10)

- EXAMPLE ⑧ RLC Networks

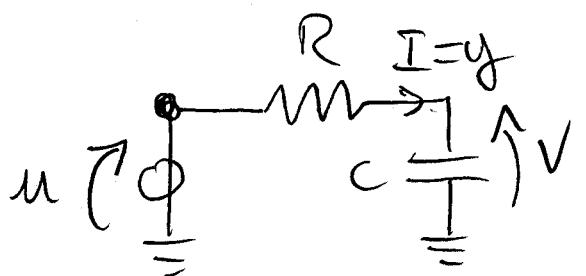


+ Kirchhoff voltage law $\sum \text{mesh } V_j = 0$

+ Kirchhoff current law $\sum \text{nodes } I_j = 0$

→ linear differential-algebraic equations

→ can be solved for $\dot{x}, y \rightarrow$ state space form



$$RI + V = u \Rightarrow I = \frac{u - V}{R}$$

$$I = C \ddot{V}$$

$$\left\{ \begin{array}{l} \dot{V} = \frac{u - V}{RC} \\ y = \frac{u - V}{R} \end{array} \right. \quad V = x$$

DYNAMICAL SYSTEMS

(11)

- REMARK

- All the shown examples are described by

$$\frac{d}{dt} (\text{stored quantity}) = \bar{z} \text{ flows}$$

Quantity	Flow
mass	mass flow
energy	power
momentum	force
angular momentum	torque
charge	current
	etc.

- The equations can be solved for \dot{x}, y

$$\dot{x}(t) = f(x(t), u(t))$$

$t \in \mathbb{R}$ time

$$y(t) = g(x(t), u(t))$$

$x \in \mathbb{R}^n$ states

state space form
(time invariant)

$u \in \mathbb{R}^m$ inputs

$y \in \mathbb{R}^p$ outputs

DYNAMICAL SYSTEMS

(12)

- TIME VARYING SYSTEMS

- In the most general case, the behaviour depends on the time also explicitly

$$\dot{x}(t) = f(x(t), u(t), t)$$

$$y(t) = g(x(t), u(t), t)$$

the same inputs have different effects @ different times

- We won't use time-varying systems in this course, so we always assume time-invariance implicitly

- STATE VARIABLES

- More general definition:

"Set of variables such that the knowledge of their values at time \bar{t} allows to disregard the input at times $t < \bar{t}$ "

- State variables \rightarrow contain all the information about past inputs of a dynamical system

DYNAMICAL SYSTEMS

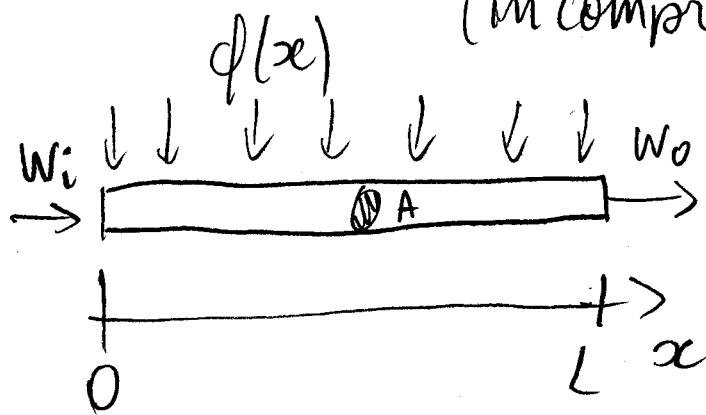
(13)

- DISTRIBUTED - PARAMETER SYSTEMS

- Lumped-parameter systems: finite number of states, inputs and outputs $v(t): \mathbb{R} \rightarrow \mathbb{R}^n$
- Distributed-parameter systems: variables are spatially distributed (1D, 2D, 3D)
Infinite-dimensional systems

$$v(x,t): \Omega \times \mathbb{R} \rightarrow \mathbb{R}$$

- EXAMPLE ① Advection equation 1D (incompressible flow)



- Rigid tube
- $A = \text{const}$
- $\rho = \text{const}$
- no thermal diffusion
- $C_p \approx C_v \approx \text{const}$
- mechanical energy neglected

- Mass balance

$$\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho A u) = 0$$

- Energy balance

$$\frac{\partial}{\partial t} (\rho A e) + \frac{\partial}{\partial x} (\rho A u h) = \varphi \cdot w$$

$$w = \rho A u$$

DYNAMICAL SYSTEMS

(14)

$$\rho = \text{const} \rightarrow \frac{\partial}{\partial t} (\rho A) = 0 \rightarrow 0 = \frac{\partial w}{\partial x}$$

$$A = \text{const} \rightarrow w(x, t) = w(t) = w_i(t) = w_u(t)$$

$$\rho = \text{const}$$

$$\text{den} dm = cdT \rightarrow \rho A_C \frac{\partial T}{\partial t} + W_C \frac{\partial T}{\partial x} = \varphi \cdot w$$

$$A = \text{const}$$

- State variables : entire set of $T(x, t)$
 $x \in \Omega \subset L$
 - Modelled by PDE \rightarrow more difficult to handle
 - However, PDEs can be approximated
 by ODEs via
 - finite differences
 - finite volumes
 - ✓ finite element
- so, even distributed-parameter systems can be approximated by
- $$\dot{x} = f(x, u)$$
- $$y = g(x, u)$$