

NOTION & STABILITY

①

- NOTION IN A DYNAMICAL SYSTEM

- Given: • a dynamical system $\dot{x}(t) = f(x(t), u(t)) \quad (1)$

$y(t) = g(x(t), u(t)) \quad (2)$

- the time history $u(t), t \in t_0^T$ of its inputs
- the initial conditions $x(0) = x_0$

- Notion:

- solution $x(t)$ of the Cauchy problem for the state equations (1)
- solution $y(t)$ of the output equations (2)

- EQUILIBRIA IN A DYNAMICAL SYSTEM

- The equilibrium is a particular motion whereby all variables are constant

$$x(t) = \bar{x} \quad y(t) = \bar{y} \quad u(t) = \bar{u}$$

- Equilibrium conditions:

$$f(\bar{x}, \bar{u}) = 0$$

$$\bar{y} = g(\bar{x}, \bar{u})$$

NOTION & STABILITY

(2)

- EXAMPLE: ELECTRICAL HEATER

$$\dot{\bar{T}}(t) = -\frac{KS}{c\eta} \bar{T}(t) + \frac{1}{c\eta} \bar{Q}_j(t) + \frac{KS}{c\eta} \bar{T}_e(t)$$

$$x = \bar{T}$$

$$u_1 = \bar{Q}_j$$

$$y_1(t) = \bar{T}(t)$$

$$u_2 = \bar{T}_e$$

$$y_2(t) = \bar{Q}_e(t) = KS [\bar{T}(t) - \bar{T}_e(t)]$$

$$y_1 = \bar{T}$$

$$y_2 = \bar{Q}_e$$

- Equilibrium conditions:

$$\begin{cases} 0 = -\frac{KS}{c\eta} \bar{T} + \frac{1}{c\eta} \bar{Q}_j + \frac{KS}{c\eta} \bar{T}_e \\ \bar{y}_1 = \bar{T} \\ \bar{y}_2 = \bar{Q}_e = KS (\bar{T} - \bar{T}_e) \end{cases}$$

$$\bar{T} = \bar{T}_e + \frac{\bar{Q}_j}{KS}$$

$$\bar{Q}_j = \bar{Q}_e = KS (\bar{T} - \bar{T}_e)$$

- System motion: assume the system is at equilibrium for $t < 0$, then \bar{Q}_j is suddenly changed at $t=0$

$$\bar{T}(t) = \bar{T}$$

$$\bar{T}(0) = \bar{T}$$

$$\bar{Q}_j(t) = \bar{Q}_j \quad (t < 0)$$

$$\bar{Q}_j(t) = \bar{Q}_j + \Delta Q \quad t \geq 0$$

$$\bar{T}_e(t) = \bar{T}_e$$

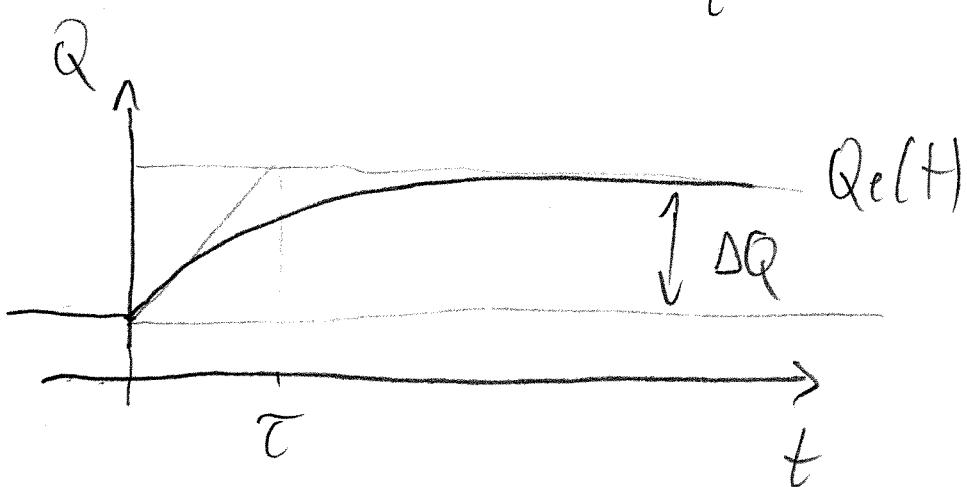
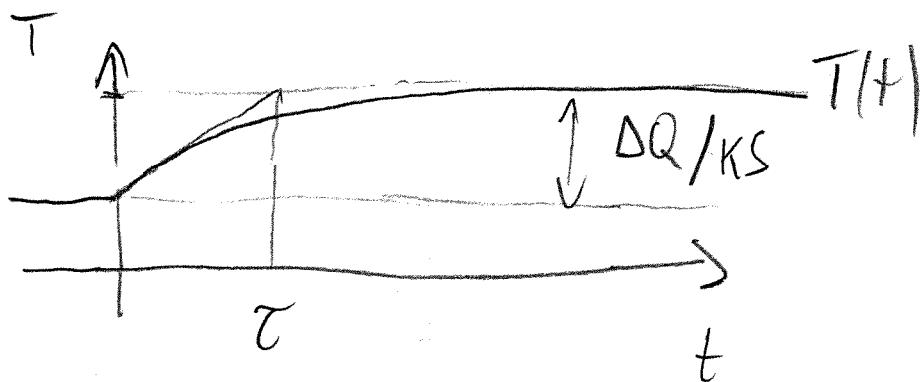
$$\bar{T}_e(t) = \bar{T}$$

MOTION & STABILITY

(3)

- The solution of the Coupling problem can be found by Euler's method for linear diff. equations

$$\left\{ \begin{array}{l} T(t) = \bar{T} + \frac{\Delta Q}{KS} \left(1 - e^{-t/\tau} \right) \quad \tau = \frac{c\theta}{KS} \\ y_1(t) = T(t) \\ y_2(t) = \bar{Q}_e + \Delta Q \left(1 - e^{-t/\tau} \right) \end{array} \right.$$



- The speed of response is governed by the parameter τ , which depends on the thermal inertia $c\theta$ and on the thermal conductance KS

MOTION & STABILITY

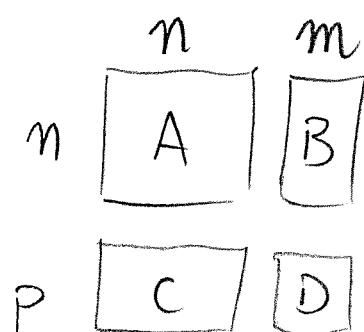
(4)

- MOTION IN LINEAR SYSTEMS

- In general, the system motion depends in a complex way from $x(0)$ and $u(t)$
- For linear systems, the dependency is simpler and superposition of effects holds
- Linear time-invariant system (LTI)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$



\dot{x} and y are linear combinations of x and u

- The motion of a LTI can be formulated by the Lagrange formula

$$x(t) = \underbrace{e^{At} x(0)}_{\text{Free motion}} + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau) d\tau}_{\text{forced motion}}$$

$$y(t) = C e^{At} x(0) + \left(\int_0^t e^{A(t-\tau)} B u(\tau) d\tau + Du(t) \right)$$

MOTION & STABILITY

(5)

- The matrix exponential is defined by extending the McLaurin expansion for scalar x

$$e^x = \sum_0^{\infty} k \frac{x^k}{k!} \rightarrow e^X = \sum_0^{\infty} k \frac{X^k}{k!}$$

- It can be proven that the series converges to a matrix containing (complex) exponential terms and that the following properties also hold

$$e^{A+B} = e^A \cdot e^B$$

$$\frac{d}{dt} e^{At} = A e^{At}$$

- Lagrange's formula can be proven by substitution to fulfill the state equations, which is unique thanks to Cauchy's theorem

$$\begin{aligned} \dot{x} &= A e^{At} + A \int_0^t e^{A(t-\tau)} B u(\tau) d\tau + \\ &\quad + \cancel{e^{A(t-t)} \overset{0}{\underset{\circ}{B}} u(t)} = \\ &= Ax(t) + Bu(t) \end{aligned}$$

MOTION & STABILITY

(6)

- SUPERPOSITION OF EFFECTS I

- The motion of a LTI is the sum of a component which only depends on the initial conditions (free motion) and of a component that only depends on the system inputs (forced motion)
- Proof: immediately follow from Lagrange's formula

- SUPERPOSITION OF EFFECTS II

- In a LTI system, if

$x_1(t)$ is the forced motion caused by $u(t) = u_1(t)$

$x_2(t)$ is the forced motion caused by $u(t) = u_2(t)$

then applying $u(t) = \alpha u_1(t) + \beta u_2(t)$ the resulting motion is $x(t) = \alpha x_1(t) + \beta x_2(t)$

- This property greatly simplifies the analysis of the motion of LTIs
- For non-linear systems, it is possible to approximate their dynamic response as that of a LTI in the neighbourhood of an equilibrium point

MOTION & STABILITY

(7)

- MOTION IN THE NEIGHBOURHOOD OF AN EQUILIBRIUM:
LINEARIZATION TECHNIQUE

- consider the LTI

$$\begin{cases} \dot{x} = f(x, u) \\ y = g(x, u) \end{cases}$$

where f, g are smooth vector functions and
 $(\bar{x}, \bar{u}, \bar{y})$ is an equilibrium point

$$0 = f(\bar{x}, \bar{u})$$

$$\bar{y} = g(\bar{x}, \bar{u})$$

- Expand the system equations by Taylor series:

$$x(t) = \bar{x} + \Delta x(t)$$

$$u(t) = \bar{u} + \Delta u(t) \quad \text{Jacobian matrices}$$

$$y(t) = \bar{y} + \Delta y(t) \quad / \quad \Downarrow$$

$$\dot{\cancel{x}} + \Delta \dot{x} = f(\bar{x}, \bar{u}) + \left. \frac{\partial f}{\partial x} \right|_{\text{eq}} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{\text{eq}} \Delta u + \text{h.o.t.}$$

$$\bar{y} + \Delta y = g(\bar{x}, \bar{u}) + \left. \frac{\partial g}{\partial x} \right|_{\text{eq}} \Delta x + \left. \frac{\partial g}{\partial u} \right|_{\text{eq}} \Delta u + \text{h.o.t.}$$

MOTION & STABILITY (8)

- If we neglect the higher-order terms, which are small in the vicinity of the equilibrium, we get an approximated LTI description

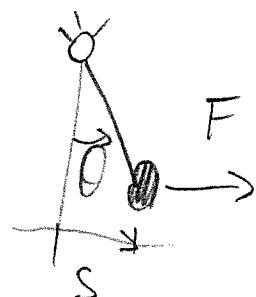
$$\dot{\delta x}(t) = \left. \frac{\partial P}{\partial x} \right|_{\text{eq}} \delta x(t) + \left. \frac{\partial P}{\partial u} \right|_{\text{eq}} \delta u(t)$$

$$\dot{\delta y}(t) = \left. \frac{\partial g}{\partial x} \right|_{\text{eq}} \delta x(t) + \left. \frac{\partial g}{\partial u} \right|_{\text{eq}} \delta u(t)$$

NB the right-hand-side terms are just the first differentials of $P(x, u)$ and $g(x, u)$

- EXAMPLE : SIMPLE PENDULUM

$$\begin{cases} \dot{\theta} = \omega \\ \ddot{\omega} = \frac{1}{ml} F \cos \theta - \frac{g}{l} \sin \theta - \frac{h}{ml^2} \omega \end{cases}$$



- Equilibria for $\bar{F} = 0$

$$\theta = \bar{\omega}$$

$$\begin{aligned} 0 &= \frac{1}{ml} \cancel{F} \cos \bar{\theta} - \frac{g}{l} \sin \bar{\theta} - \cancel{\frac{h}{ml^2} \bar{\omega}} \\ \bar{y} &= l \sin \bar{\theta} \end{aligned}$$

MOTION & STABILITY

(g)

- Solution:

$$\bar{\omega} = 0$$

$$\bar{\theta} = k\pi, \quad k=0, \pm 1, \pm 2, \dots$$

- In fact, there are just 2 interesting equilibria

$$\bar{x}_A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$\bar{x}_B = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$



- Linearized equations:

$$\left\{ \begin{array}{l} \dot{\theta} = \omega \\ \dot{\omega} = \left(-\frac{1}{ml} F \sin \bar{\theta} - \frac{g}{l} \cos \bar{\theta} \right) \omega - \frac{h}{ml^2} \omega + \frac{\cos \bar{\theta}}{ml} DF \\ \dot{y} = l \cos \bar{\theta} \omega \end{array} \right.$$

(right-hand-sides computed as 1st differentials)

MOTION & STABILITY

(10)

- Linearized equations around \bar{x}_A

$$\begin{cases} \ddot{\Delta\theta} = \Delta w \\ \dot{\Delta w} = -\frac{g}{l} \Delta\theta - \frac{h}{ml^2} \Delta w + \frac{1}{ml} \Delta u \\ \Delta y = l \Delta\theta \end{cases}$$

formally (*)
analogous
to spring-mass
system equations!

- Linearized equations around \bar{x}_B

$$\begin{cases} \ddot{\Delta\theta} = \Delta w \\ \dot{\Delta w} = (+) \frac{g}{l} \Delta\theta - \frac{h}{ml^2} \Delta w (-) \frac{1}{ml} \Delta u \\ \Delta y = (-) l \Delta\theta \end{cases}$$

- different kinematics
- repulsive force!

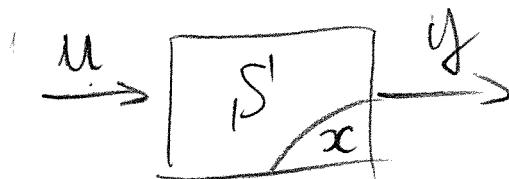
$$(*) -\frac{k}{m} \Leftrightarrow -\frac{g}{l}$$

- The small motion of openpendulum close to \bar{x}_A is analogous to a spring-mass system in particular, for $h \rightarrow 0$ the period of oscillation is independent of the amplitude a principle discovered empirically by Galileo Galilei before the year 1592

MOTION & STABILITY

(11)

- STABILITY OF EQUILIBRIA



- Consider a dynamical system S and a particular equilibrium point $(\bar{x}, \bar{u}, \bar{y})$
- The stability of the equilibrium concerns the behaviour of the system motion after a small perturbation has been applied
- There are two alternative definitions of stability
 - ① Internal stability : perturb $x(0)$ with constant $u(t)$ and look at the motion of $x(t)$
 - ② External stability : perturb $u(t)$ without touching $x(0)$ and look at the motion of $y(t)$
- In many cases the two definitions are equivalent
in others they aren't - a thorough discussion is beyond the scope of this course
- In the following we focus on internal stability

NOTION & STABILITY

(12)

- DIRAC'S DELTA (OR IMPULSE) FUNCTION

- Rectangular pulse function

$$\delta_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \leq t \leq \epsilon \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta_\epsilon(t) dt = 1 \quad \text{for sufficiently large } \alpha > 0$$

- Dirac's delta function $\delta(t)$

- can be intuitively defined as the "limit" of $\delta_\epsilon(t)$ as $\epsilon \rightarrow 0$

- Requires the concept of distributions (a generalization of functions) to be formally defined

- Properties of $\delta(t)$

- $\delta(t) = 0 \quad t \neq 0$

- $\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad \forall \alpha > 0$

MOTION & STABILITY

(13)

- It can be used to model physical phenomena having impulsive behaviour, i.e., short duration and interest on their time integrals
- Example: 1D collision

$$m \dot{v}(t) = F(t)$$

$$\int_{t_{\text{before}}}^{t_{\text{after}}} m \dot{v}(t) dt = \int_{t_{\text{before}}}^{t_{\text{after}}} F(t) dt$$

$$m[v(t_{\text{after}}) - v(t_{\text{before}})] = \int_{t_{\text{before}}}^{t_{\text{after}}} F(t) dt$$

if the main interest is the change in speed and not, e.g., the maximum value of $F(t)$, then one can approximate $F(t)$ by an impulse

$$F(t) = m \Delta v \delta(t)$$

MOTION & STABILITY

(14)

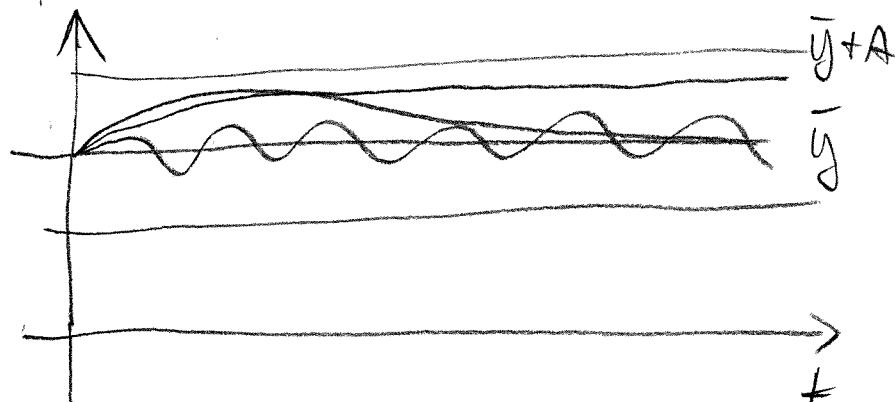
- EXTERNAL STABILITY OF EQUILIBRIA

- Let \bar{x}, \bar{y} be equilibrium values of input and output of a dynamical system

- Apply $u(t) = \bar{u} + \varepsilon \cdot \delta(t)$

① If $HA > 0$ $\exists \alpha > 0$ such that

$|\varepsilon| < \alpha \Rightarrow |y(t) - \bar{y}| < A \quad \forall t$, then the equilibrium is stable

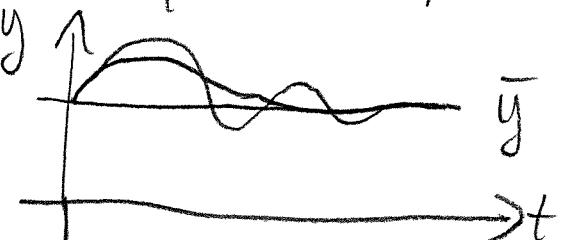


it is possible to keep the output(s) arbitrarily close to the equilibrium value by applying a small enough perturbation to the input(s)

② If all the above conditions hold, and also

$\lim_{t \rightarrow \infty} |y(t) - \bar{y}| = 0$, then the equilibrium

is asymptotically stable

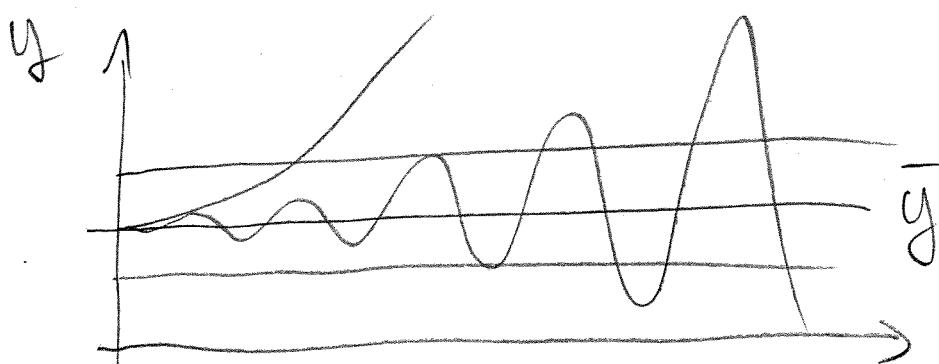


NOTION & STABILITY

(15)

③ In all other cases, the equilibrium is unstable

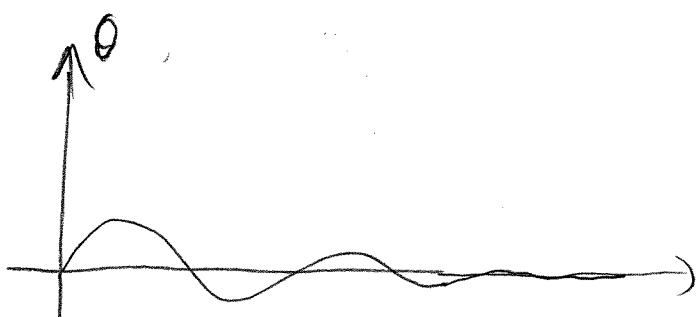
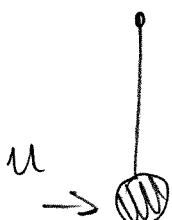
④ If ① holds but ② doesn't, then the equilibrium is simply stable



It is not possible to stay close to the equilibrium, no matter how small the perturbation $\epsilon \cdot \delta(t)$ is

- EXAMPLE: SIMPLE PENDULUM

- Lower equilibrium with friction

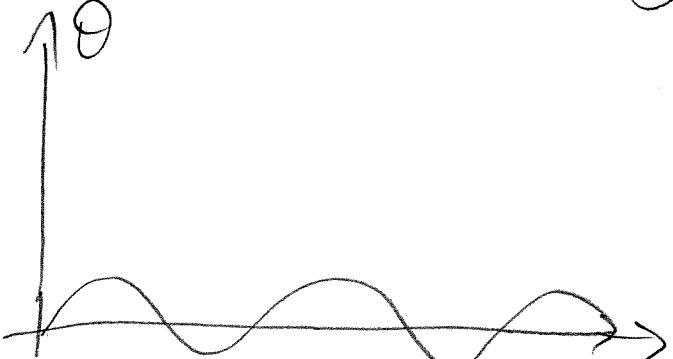
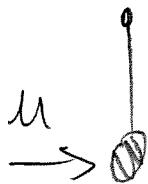


asymptotically stable
equilibrium

MOTION & STABILITY

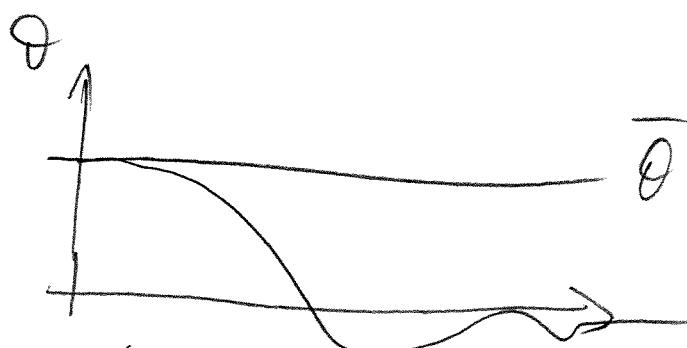
(16)

- Lower equilibrium
no friction



simply stable equilibrium

- Upper equilibrium



unstable equilibrium

- STABILITY IN LTI SYSTEMS

- Given a LTI dynamical system S , it can be proven by the superposition principle that all its equilibria share the same stability properties
- Therefore, we (somewhat improperly) refer to stable/unstable systems rather than just equilibria
- This only makes sense for LTI systems