Control Systems

(Prof. Casella)

Written Exam – July 2^{nd}, 2015

Answer Sheet

Question 1

With reference to feed-back control systems, explain why the input-output dynamic behaviour of the plant to be controlled is crucial for the system performance. Also explain why, on the other hand, a very precise knowledge of that dynamic behaviour is not required, if the controller is designed properly.

The input-output dynamics of the plant is a key factor in determining the dynamic behaviour of the closed-loop system, first and foremost stability, then response speed and ability to closely follow the set point and reject disturbances. A controller which is not correctly designed for a certain plant dynamics can make the closed-loop system unstable.

On the other hand, if the controller is designed with a good phase margin, significant but yet small changes in the plant dynamics will not alter its stability, and lead to changes of closed-loop dynamic features such as response time, settling time or damping of oscillations that are acceptable in most use cases.

Question 2

Briefly explain why the stability of a linear system with transfer function $G(s)$ is completely determined by its poles. Then, state precisely under which conditions such system will be asymptotically stable, simply stable, or unstable.

The external stability of a linear system is determined by its impulse response, in particular by the fact that it is bounded or unbounded, and that it does or does not approache zero as time approaches infinity. Now, the impulse response is the inverse Laplace transformation of $G(s)$, which can be decomposed in the summation of simple rational functions whose poles are the poles of $G(s)$, and whose inverse transforms are the components of the impulse response. Therefore, the stability is determined by the poles of $G(s)$.

The system is asymptotically stable if all poles have negative real part. If all poles have negative or zero real part, and the ones which have zero real part have multiplicity one, then the system is simply stable. Otherwise, the system is unstable.

Question 3

Consider a one-dimensional spring-mass mechanical system, in which a body of mass M is connected to the ground by a spring and by a damper, and subject to an external force F. Contrary to a standard spring-mass harmonic oscillator, the spring has a nonlinear force-displacement characteristic, where the force grows more than proportionally to the spring deformation.

The equations describing the system are the following, where x is the body displacement, v is the body velocity, F_s is the spring force, F_d is the damper force, K and x_0 are the spring parameters, and h is the damper friction coefficient.

$$M\ddot{x}=F_s+F_d+F$$
$$F_s=-\frac{K}{x_0^2}x\left(x^2+x_0^2\right)$$
$$F_d=-hv$$

3.1 Write down the state and output equations in standard state-space form, considering F as input and x, v as outputs

The system is similar to a standard harmonic oscillator. The second order differential equation can be turned into two first-order explicit differential equations by introducing an extra differential equation relating the position and the velocity

$$\dot{x}=v$$
$$\dot{v}=\frac{1}{M}\left(-\frac{K}{x_0^2}x\left(x^2+x_0^2\right)-hv+F\right)$$
$$y_1=x$$
$$y_2=v$$

3.2 Compute the equilibrium conditions for the system

$$\bar{v}=0$$
$$\frac{K}{x_0^2}\bar{x}\left(\bar{x}^2+x_0^2\right)=\bar{F}$$

3.3 Write down the system's linearized equations around a generic equilibrium

$$\Delta\dot{x}=\Delta v$$
$$\Delta\dot{v}=\frac{1}{M}\left(-K\,\Delta x-3K\frac{\bar{x}^2}{x_0^2}\Delta x-h\,\Delta v+\Delta F\right)$$
$$\Delta y_1=\Delta x$$
$$\Delta y_2=\Delta v$$

3.4 Assuming the equilibrium value of F is zero, compute the transfer functions of the system between the deviations of the input ΔF and the deviations of the outputs Δx and Δv.

$$\frac{K}{x_0^2} x \left(x^2 + x_0^2 \right) = 0 \Rightarrow x = 0$$

$$\Delta x = \frac{1}{Ms^2 + hs + K} \Delta F$$

$$\Delta y = \frac{s}{Ms^2 + hs + K} \Delta F$$

3.5 Determine under which conditions the system shows an oscillatory behaviour in response to a unit step change of the input ΔF, then plot the qualitative diagrams of the corresponding $\Delta x(t)$ and $\Delta v(t)$.

Oscillatory behaviour takes place when the transfer function has complex poles, which happens if $h < 4KM$.

The diagram of $\Delta x(t)$ is the standard step response of a transfer function with two complex poles; assuming $h > 0$, the final value is $1/K$.

$\Delta v(t)$ is the derivative of $\Delta x(t)$, so the pseudo-period of the oscillations and the settling time are the same, while the final value is zero.

3.6 Assume now that the equilibrium value of F is no longer zero, but rather that the spring is pre-loaded by a suitable force value such that the corresponding equilibrium displacement is x_0. Explain how the plots determined at the previous point change, all other parameters being unchanged.

In this case, the parameter which multiplies $\Delta x(t)$ in the second linearized state equation becoms $4K$ instead of K; in other words, the spring become four times stiffer.

As a consequence, the gain of both functions is reduced by a factor four, and so is the final value of the response of Δx. The natural frequency of the poles is doubled, so the pseudo-period of the responses is approximately halved. On the other hand, the damping coefficient ξ is also halved, so the settling time remains the same.

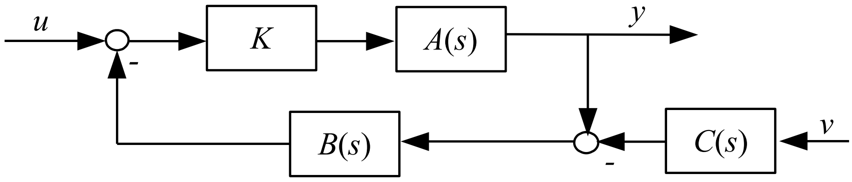
Question 4

Considering the following block diagram

$$A(s)=\frac{10}{1+s}$$

$$B(s)=\frac{1}{1+s}$$

$$C(s)=\frac{5}{1+s}$$



4.1 Compute the transfer functions between the inputs *u* and *v* and the output *y*.

$$Y(s)=G_u(s)U(s)+G_v(s)V(s)$$

$$G_u(s)=\frac{K\,A(s)}{1+K\,A(s)B(s)}=\frac{10K(1+s)}{s^2+2s+1+10K}$$

$$G_v(s)=\frac{K\,A(s)B(s)C(s)}{1+K\,A(s)B(s)}=\frac{50K}{(1+s)(s^2+2s+1+10K)}$$

4.2 Determine for which values of the parameter *K* the system shows exponentially diverging oscillations in response to step changes of the inputs.

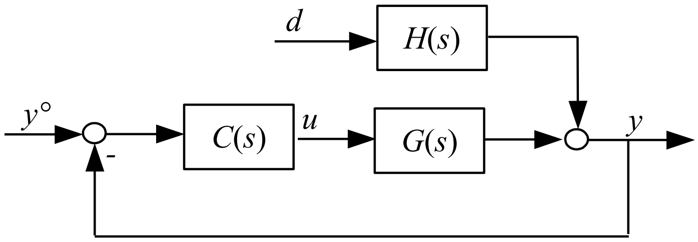
The requested behaviour requires complex conjugate poles with positive real part. The condition for having complex conjugate poles is 1+10K > 1; however, under this condition, the real part of the poles is –1/2(1+10K) < 0. Therefore, there is no value of *K* that leads to exponentially diverging oscillations.

Question 5

Consider the following control system, where the unit of time constants is the second:

$$G(s)=\frac{500}{s(1+100s)(1+30s)(1+10s)(1+5s)}$$

$$H(s)=\frac{10}{s(1+100s)}$$



5.1 Design a PI or PID controller with a bandwidth of 0.003 rad/s and at least 60° phase margin.

At the specified crossover frequency, the frequency response $G(j\omega)$ has a phase of -114°. The specification requires the crossover phase to be not less then -120°, so the controller must have a phase of less than -6° at crossover. This can be achieved by a PI controller with $T_i = 10/\omega_c = 3330$ s. The bandwidth requirement is obtained by setting $K_p = 6{\cdot}10^{-6}$.

In order to avoid the very long settling time of $5T_i$ of the disturbance response, it is possible to introduce some derivative action, which allows to reduce T_i to smaller values (thus reducing the disturbance response settling time) while still meeting the requirement on the phase margin.

5.2 Design a PID controller with the highest bandwidth you can get for a phase margin of 45°, assuming $N = 10$ for the derivative action.

Assuming $T_i >> T_d$, the loop transfer function can be approximated by:

$$L(s)=K_p\frac{(1+sT_i)(1+sT_d)}{sT_i(1+s\frac{T_d}{10})}\frac{500}{s(1+100s)(1+30s)(1+10s)(1+5s)}$$

If one uses the second zero of the controller to cancel the second pole of the process, taking $T_d = 100$, then the phase margin is

$$\varphi_m=180°-90°+atan(\omega_c T_i)-atan(10\omega_c)-90°-atan(30\omega_c)-atan(10\omega_c)-atan(5\omega_c)$$

The first arctangent can reach values very close to 90° by taking very large values of T_i , which is however not advisable, unless one wants to accept extremely long disturbance settling times. Assuming one accepts $T_i = 10/\omega_c$, then

$$\varphi_m=84°-atan(10\omega_c)-atan(30\omega_c)-atan(10\omega_c)-atan(5\omega_c)=45°$$

which can be solved approximately by trial and error, giving $\omega_c = 0.0125$ and $T_i = 800$.

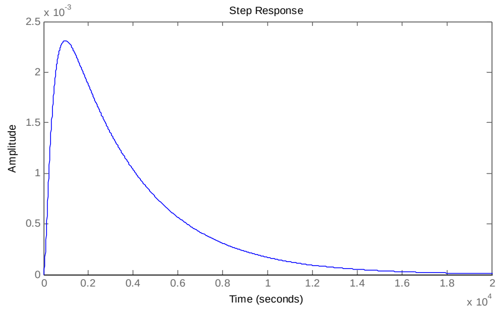
The found value of the crossover frequency is obtained by setting $K_p = 2.5{\cdot}10^{-5}$.

5.3 Plot the qualitative diagrams of the response of the controlled output y to a step change in the disturbance d for the two controllers designed at points 5.1 and 5.2.

The transfer function between d and u is $H(s)S(s)$, where the sensitivity transfer function $S(s)$ can be estimated remembering that its frequency response is approximately $1/L(j\omega)$ for $\omega \ll \omega_c$, 1 for $\omega \gg \omega_c$, and that there is a real pole at about ω_c if $\phi_m \geq 60°$, or a pair of conjugate poles with $\omega_n \approx \omega_c$, $\xi \approx \phi_m \geq/ 100°$ and a zero at about ω_c if $\phi_m < 60°$.

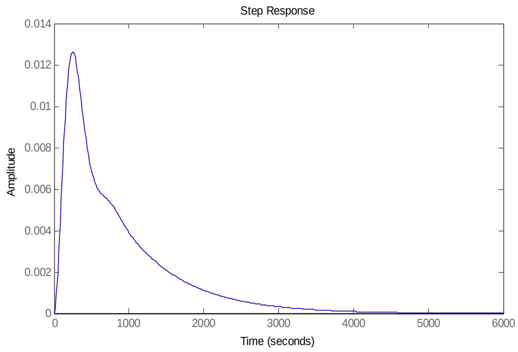
$$S_1(s) \approx \frac{s^2 T_i/\omega_c}{(1+sT_i)(1+s/\omega_c)}$$

$$H(s)S_1(s) \approx \frac{10s}{(1+3330s)(1+333s)(1+100s)}$$



$$S_2(s) \approx \frac{s^2 T_i/\omega_c(1+\frac{s}{\omega_c})}{(1+sT_i)(1+2\cdot 0.45 s/\omega_c+\frac{s^2}{\omega_c^2})}$$

$$H(s)S_2(s) \approx \frac{10s}{(1+800s)(1+72s+6400s^2)}$$



Note that in the second case the oscillations are well damped and have a pseudo-period of 550 s, which is much faster than the settling time of 4000 s of the dominant pole, so they are barely noticeable.

5.4 Determine the asymptotic amplitude of the oscillations of the manipulated variable u corresponding to a disturbance $d = \sin(2t)$ for the two controllers designed at points 5.1 and 5.2.

The transfer function between d and u is $H(s)C(s)$. The frequency of the disturbance is much higher than the crossover frequency, so in both cases $Q(j\omega) = C(j\omega)$.

For a PI controller, the high-frequency gain is K_p, while for a real PID it is K_pN, while $|H(j2)|=0.025$.

The asymptotic amplitude in case 5.1 is therefore $1.5\cdot 10^{-7}$, while it is $6.25\cdot 10^{-6}$ in case 5.2.