Control Systems

(Prof. Casella)

Written Exam – July 22nd, 2015

Answer Sheet

Surname:...

Name: ....

Reg. Number:...............................

Signature:.....................................

Notices:

- This booklet is comprised of 7 sheets – Check that it is complete and fill in the cover.
- Write your answers in the blank spaces with short arguments, including only the main steps in the derivation of the results.
- You are not allowed to leave the classroom unless you hand in the exam paper or withdraw from the exam.
- You are not allowed to consult books or lecture notes of any kind.
- Please hand in only this booklet at the end of the exam – no loose sheets.
- The clarity and order of your answers will influence how your exam is graded.

Question 1

Define the frequency response of a linear, time-invariant system. Then, state how it can be used to determine the system response to a sinusoidal input in the time domain.

If the system has transfer function $G(s)$, then the frequency response of the system is the complex function $G(j\omega)$, where the real frequency ω goes from 0 to infinity.

If the system is asymptotically stable, then the asymptotic response of the system's output to a sinusoidal input $u(t) = A\, cos(\omega t + \phi)$ is $y(t) = |G(j\omega)| \cos(\omega t + \phi + \arg(G(j\omega)))$.

Question 2

Explain the phenomenon of integral wind-up in linear controllers and how it can be avoided.

See lecture notes.

Question 3

Consider a thermo-hydraulic system in which a volume V of well-mixed fluid exchanges heat with a wall that is kept at a constant temperature T_w by some external means. A mass flow rate w_i of fluid at temperature T_i enters the volume, and a mass flow rate of fluid w_o leaves it. The heat transfer coefficient between the fluid and the wall is proportional to $w_o^{0.8}$. It is assumed for simplicity that the fluid has constant density and that $dh = de = c\,dT$.

The equations describing the system are the following, where M is the constant fluid mass, T the fluid temperature, c the constant fluid specific heat capacity, Q the heat flow to the fluid, and w_{nom} the nominal value of the flow rate:

$$Mc\frac{dT}{dt}=w_i c T_i - w_o c T + Q$$
$$w_i = w_o$$
$$Q = UA(T_w - T)$$
$$UA = UA_{nom}\left(\frac{w_o}{w_{o,nom}}\right)^{0.8}$$

3.1 Write down the state and output equations in standard state-space form, considering w_i and T_i as inputs, T and Q as outputs

$$\frac{dT}{dt}=\frac{1}{M}\left[w_i(T_i-T)+\frac{UA_{nom}}{c}\left(\frac{w_i}{w_{nom}}\right)^{0.8}(T_w-T)\right]$$
$$y_1 = T$$
$$y_2 = Q = UA_{nom}\left(\frac{w_i}{w_{nom}}\right)^{0.8}(T_w-T)$$

3.2 Compute the equilibrium conditions for the system, assuming $w_o = w_{nom}$

$$\bar{w}_i c(\bar{T}-\bar{T}_i)=\bar{Q}$$
$$\bar{Q}=UA_{nom}(\bar{T}_w-\bar{T})$$
$$\bar{T}-\bar{T}_i=\frac{UA_{nom}}{\bar{w}_i c}(\bar{T}_w-\bar{T})$$

3.3 Write down the system's linearized equations around the previously found equilibrium

$$\frac{d\Delta T}{dt}=\frac{1}{M}\left[\bar{w}_i\Delta T_i-\bar{w}_i\left(1+\frac{UA_{nom}}{\bar{w}_i c}\right)\Delta T+\Delta w_i\left(\bar{T}_i-\bar{T}+0.8\frac{UA_{nom}}{\bar{w}_i c}(\bar{T}_w-\bar{T})\right)\right]$$
$$\Delta y_1=\Delta T$$
$$\Delta y_2=\Delta Q=-UA_{nom}\Delta T+0.8\frac{UA_{nom}}{\bar{w}_i}(\bar{T}_w-\bar{T})\Delta\bar{w}_i$$

$$\frac{d\Delta T}{dt}=\frac{1}{M}\left[\bar{w}_i\Delta T_i-\bar{w}_i\left(1+\frac{UA_{nom}}{\bar{w}_i c}\right)\Delta T-\Delta w_i\frac{0.2\bar{Q}}{\bar{w}_i c}\right]$$
$$\Delta y_1=\Delta T$$
$$\Delta y_2=\Delta Q=-UA_{nom}\Delta T+0.8\frac{\bar{Q}}{\bar{w}_i}\Delta\bar{w}_i$$

3.4 Compute the transfer functions of the system between the deviations of the input ΔT_i and the deviations of the outputs ΔT and ΔQ and write them down in gain / time constant form.

$$\Delta T = \frac{1}{1+\frac{UA_{nom}}{\bar{w}_i c}} \frac{1}{1+s\tau} \Delta T_i$$

$$\Delta Q = -\frac{\frac{UA_{nom}}{1}}{1+\frac{UA_{nom}}{\bar{w}_i c}} \frac{1}{1+s\tau} \Delta T_i = -\frac{\bar{w}_i c\, UA_{nom}}{\bar{w}_i c + UA_{nom}} \frac{1}{1+s\tau} \Delta T_i$$

$$\tau = \frac{M}{\bar{w}_i} \frac{1}{1+\frac{UA_{nom}}{\bar{w}_i c}} = \frac{Mc}{\bar{w}_i c + UA_{nom}}$$

3.5 Plot the qualitative diagrams of the unit step response of the transfer functions computed at point 3.4

Standard plots of first-order systems with one negative real pole and no zeros

3.6 Assume $\Delta T_i = sin(\omega t)$. Determine for which values of ω the corresponding oscillations of ΔT have an amplitude which is much smaller than the final value of the response to $\Delta T_i = step(t)$

The final value of the step response is $G(0)$, while the amplitude of the sinusoidal oscillations is $G(j\omega)$. The latter is smaller than the former if $\omega >> 1 / \tau$.

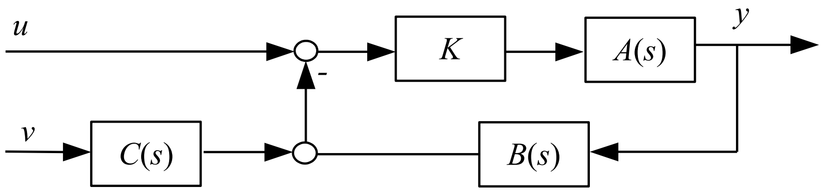
Question 4

Considering the following block diagram

$$A(s)=\frac{100}{s}$$

$$B(s)=\frac{1}{1+s}$$

$$C(s)=\frac{40}{1+10s+10s^2}$$



4.1 Compute the transfer functions between the inputs u and v and the output y.

$$Y(s)=\frac{K\,A(s)}{1+KA(s)B(s)}U(s)-\frac{K\,A(s)C(s)}{1+KA(s)B(s)}V(s)$$
$$Y(s)=\frac{100\,K\,(1+s)}{s^2+s+100\,K}U(s)-\frac{4000\,K\,(1+s)}{(s^2+s+100\,K)(1+10s+10s^2)}V(s)$$

4.2 Determine for which values of the parameter K the system shows damped, exponentially decaying oscillations in response to step changes of the inputs.

In order to show damped, asymptotically decaying oscillations, the transfer functions need to have complex poles with negative real part, and they need to be asymptotically stable.

The polynomial $1 + 10s + 10s^2$ has two negative real roots, so it satisfies the stability requirement, but cannot produce oscillations. As to the polynomial $s^2 + s + 100K$, asymptotic stability requires $K > 0$, while complex poles are obtained if $\Delta < 0$ (hence $K > 1/400$).

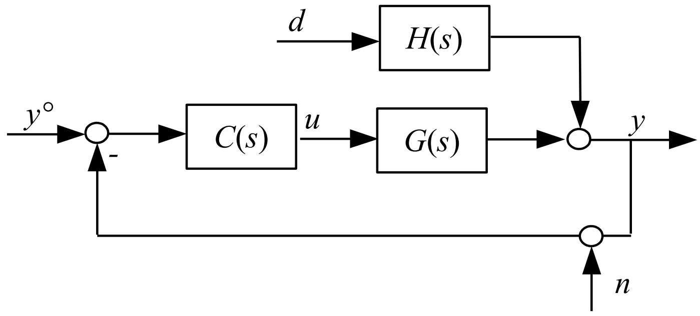
Summing up, $K > 1/400$.

Question 5

Consider the following control system, where the unit of time constants is the second:

$$G(s) = \frac{500(1-10s)}{s(1+20s)}$$

$$H(s) = \frac{150}{s}$$



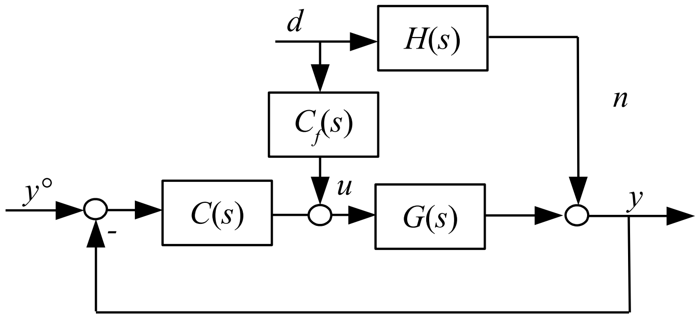
5.1 Design a PI or PID controller with a bandwidth of 0.01 rad/s and at least 50° phase margin.

A PI controller with $K_p = 2 \cdot 10^{-5}$ and $T_i = 500$ has the required crossover frequency and a phase margin of 62°.

5.2 Plot the qualitative diagrams of the response of the controlled output *y* to a step change of the set point *y*°.

A first approximation is just a first-order system response with a time constant of 100 s and a settling time of 500s. In fact, the right-half-plane zero in *G*(*s*) also shows up in the complementary sensitivity function, so the step response of *y* has a small undershoot at the beginning of the transient.

5.3 Design a feed-forward disturbance compensator for the system. Is there any upper bound to the bandwidth where the compensator can be effective?



The ideal disturbance compensator has transfer function:

$$C_f(s) = -\frac{H(s)}{G(s)} = -\frac{3}{10}\frac{1+20s}{1-10s}$$

which cannot be implemented because of the unstable pole. Low-frequency approximations can be used, such as the static compensator -3/10. However, they can only be effective in a range of frequencies lower than the frequency of the unstable pole, i.e., 0.1 rad/s.

5.4 How is the dynamic performance of the system affected if the gain of G(s) turns out to be 1000 instead of 500?

The bandwidth is doubled, while the phase margin is reduced to about 51°. The response has now some well-damped oscillations (the damping coefficient is about 0.5), and the settling time is more or less the same as before.

5.5 Assume $n = sin(10t)$. Is it possible to modify the controller designed at point 5.1 so that the set point tracking and the rejection of the disturbance *d* is unchanged, but the amplitude of the oscillations of the manipulated variable *u* is drastically reduced?

The frequency of the disturbance *n* is much higher than the crossover, so that the frequency response function $-Q(j10)$ between *n* and *u* can be approximated by just $-C(j10) = K_p$.

It is possible to meet the requirement by adding a low-pass filter to the controller so that the magnitude of $C(j10)$ is greatly reduced, while the phase and magnitude of $C(j\omega)$ are not significantly changed. This is possible by setting the frequency of the poles of the filter at a frequency much higher than 0.01 but significantly less than 10. For example, the following controller reduces the amplitude of the control oscillations by a factor 100, while only reducing the phase margin by 1° only

$$C(s) = K_p \frac{1+sT_i}{sT_i} \frac{1}{(1+s)^2}$$